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Testing the Witten–Veneziano mechanism with the Yang–Mills gradient flow on the lattice

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Outline

Theoretical introduction
- The $U(1)_A$ problem
- The Witten–Veneziano mechanism

Lattice
- Lattice regularization
- The Yang–Mills gradient flow

Results
- Thermodynamic limit
- Continuum limit

Conclusions

The $U(1)_A$ problem

- The η' meson is (almost) a flavour singlet combination of light quarks
- No parity partner in nature $\Rightarrow U(1)_V \times U(1)_A$ broken to $U(1)_V$
- The η' is too heavy to be the ninth pseudo Nambu–Goldstone boson: $m_{\eta'} < \sqrt{3} m_\pi$ [Weinberg 1975]

Pseudoscalar mesons

Meson	Quark content	Mass [MeV]
π^+	$u\bar{d}$	139.570 18(35)
π^0	$\frac{d\bar{d}-u\bar{u}}{\sqrt{2}}$	134.9766(6)
π^-	$d\bar{u}$	139.570 18(35)
K^+	$u\bar{s}$	493.677(16)
K^0	$d\bar{s}$	497.614(24)
$\bar{K}^0$	$s\bar{d}$	497.614(24)
K^-	$s\bar{u}$	493.677(16)
η	$\cos\theta\eta_8 + \sin\theta\eta_0$	547.862(18)
η'	$-\sin\theta\eta_8 + \cos\theta\eta_0$	957.78(6)

$$\eta_8 = \frac{d\bar{d} + u\bar{u} - 2s\bar{s}}{\sqrt{6}} \qquad \theta \simeq -11.4°$$

$$\eta_0 = \frac{d\bar{d} + u\bar{u} + s\bar{s}}{\sqrt{3}}$$

The $U(1)_A$ problem: Why is the η' much heavier than other pseudo Nambu–Goldstone bosons?

Some definitions

Yang–Mills theory Euclidean action admits a θ-term

$$S[A] = \int \mathrm{d}^4x \, \frac{1}{4g^2} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta q(x)$$

where the topological charge density

$$q(x) = \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

- Locally, is the divergence of a current $q(x) = \partial_\mu K_\mu(x)$
- Its spacetime integral $Q = \int \mathrm{d}^4x q(x) \in \mathbb{Z}$ on classical configurations of finite Euclidean action
- $U(1)_A$ is broken by an anomaly $\propto 2N_f q(x)$

The topological susceptibility is the two-point function of $q(x)$ (at zero momentum)

$$\chi_t = \int \mathrm{d}^4x \, \langle q(x) q(0) \rangle$$

The Witten–Veneziano mechanism

A mechanism to solve the $U(1)_A$ problem based on the vanishing of the anomaly in the large N_c limit

[Witten 1979; Veneziano 1979]

- In the $N_c \to \infty$ limit $U(1)_A$ is restored. The η' is a Nambu–Goldstone boson
- Assuming leading order dependence on θ in Yang–Mills theory, $\chi_t^{\mathrm{YM}} \neq 0$ and $\mathcal{O}(1)$ in large N_c
- Dynamical quarks are an $\mathcal{O}\left(\frac{1}{N_c}\right)$ effect
- But no θ dependence in QCD with massless quarks $\Rightarrow \chi_t = 0$

The η' gets a mass $M^2_{\eta'} = \mathcal{O}\left(\frac{1}{N_c}\right)$ given by the Witten–Veneziano formula

$$M^2_{\eta'} = \frac{2N_f}{F^2}\chi_t^{\mathrm{YM}} + \mathcal{O}\left(\frac{1}{N_c^2}\right)$$

The dilute instanton gas

A solution of the $U(1)_A$ problem proposed by 't Hooft using instantons: dilute instanton gas approximation ['t Hooft 1976]

- Semiclassical approximation
- Free energy dependence on θ

$$F(\theta) = -\ln Z(\theta) = -VA(\cos\theta - 1)$$

- Arbitrary normalization $A \Rightarrow$ no prediction for χ_t^{YM}

$$\chi_t^{\mathrm{YM}} = \frac{1}{V}\langle Q^2 \rangle = \frac{1}{V}\left.\frac{\mathrm{d}^2 F}{\mathrm{d}\theta^2}\right|_{\theta=0} = A$$

- No IR bound on instantons size
- Expected to be valid at high temperature

Higher moments

We need to study higher moments of the probability distribution of the topological charge Q in Yang–Mills theory

- Higher cumulants are obtained deriving $F(\theta)$

$$\left\langle Q^{2n} \right\rangle^{\text{con}} = (-1)^{n+1} \frac{\mathrm{d}^{2n}}{\mathrm{d}\theta^{2n}} F(\theta) \bigg|_{\theta=0}$$

- We define the ratio between the 4th and the 2nd cumulant

$$R \equiv \frac{\left\langle Q^4 \right\rangle^{\text{con}}}{\left\langle Q^2 \right\rangle} = \frac{\left\langle Q^4 \right\rangle - 3\left\langle Q^2 \right\rangle^2}{\left\langle Q^2 \right\rangle}$$

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’t Hooft

$$R = \frac{\langle Q^4\rangle^{\mathrm{con}}}{\langle Q^2\rangle} = 1$$

Witten & Veneziano

$$R = \frac{\langle Q^4\rangle^{\mathrm{con}}}{\langle Q^2\rangle} \propto \frac{1}{N_c^2}$$

Lattice regularization

- Euclidean YM theory is discretized on a four-dimensional lattice with lattice spacing a. The lattice action
  - reduce to the action in the continuum for $a \to 0$
  - is gauge invariant at finite lattice spacing
- The gauge field on the lattice is defined on links between lattice sites: $U_\mu(x) \in SU(3)$
- The field strength tensor is defined with plaquettes
- Wilson plaquette action

$$S_W[U] = \frac{2N_c}{g^2} \sum_P \left(1 + \frac{1}{2N_c} \mathrm{tr}\left\{ U_P + U_P^\dagger \right\} \right)$$

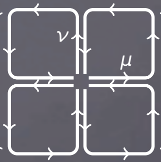
Topological charge on the lattice

Naïve discretization of the topological charge density

$$q(x) = \frac{1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$$

- $F^a_{\mu\nu}(x)$ given in terms of plaquettes (clover definition)

$$F^a_{\mu\nu}(x) = \frac{1}{4}P^a\Bigg(\qquad\Bigg)$$



with $P^a(U)$ projecting over $\mathfrak{su}(3)$ Lie algebra

Topological charge on the lattice

Naïve discretization of the topological charge density

$$q(x) = \frac{1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$$

- $F^a_{\mu\nu}(x)$ given in terms of plaquettes (clover definition)
- Two-point function at same point $\langle q(x)q(x)\rangle$ has contact terms $\Rightarrow$ additive renormalization

Solutions:

- Fermionic definition, using a Dirac operator satisfying Ginsparg–Wilson, as Neuberger–Dirac operator D_N

$$q(x) = \frac{1}{2a^3}\operatorname{tr}\gamma_5 D_N(x,x)$$

  Index theorem: $Q(x) = \sum q(x) = \frac{a}{2}\operatorname{tr}\gamma_5 D_N(x,x) = n_- - n_+$

  [Giusti, Rossi, Testa, Veneziano 2002; Giusti, Rossi, Testa 2004; Lüscher 2004]
- or...

The Yang–Mills gradient flow

The gradient flow is the solution of the initial value problem [Lüscher 2010]

$$\dot{V}_\mu(t,x) = -g^2\left\{\partial_{\mu,x} S_W[V(t)]\right\} V_\mu(t,x) \qquad V_\mu(0,x) = U_\mu(x)$$

- $V_\mu(0,x)$: gauge field configurations from Monte-Carlo
  $\Downarrow$ evolving the 'flow-time' t with gradient flow
  $V_\mu(t,x)$: configurations smoothed within a radius $\sqrt{8t}$
- We can use a naïve discretization of the topological charge density in the continuum, applied to $t > 0$ configurations [Lüscher, Weisz 2011]

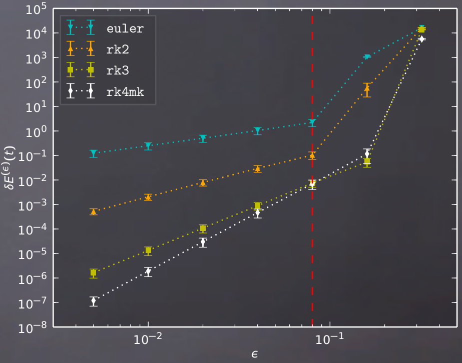
$$q(t,x) = \frac{1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(t,x) F^a_{\rho\sigma}(t,x)$$

- Numerical integration of gradient flow is less expensive than constructing the Neuberger-Dirac operator

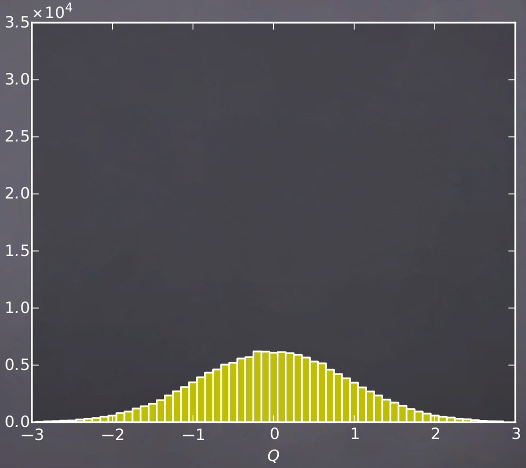
Runge–Kutta integrator

To numerically solve the gradient flow equation, we implemented a fourth-order Runge–Kutta–Munthe-Kaas method

- Structure-preserving method: $SU(3)$ Lie group structure is exactly preserved
- Very small systematic errors from numerical integration, negligible with respect to statistical errors

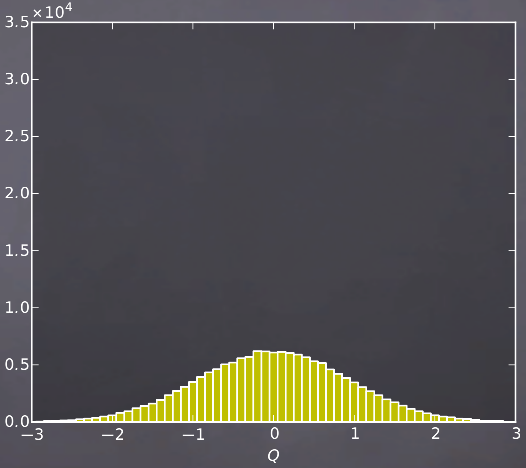


Topological charge distribution



at flow-time $t = 0.0000\,\mathrm{fm}^2$

Topological charge distribution



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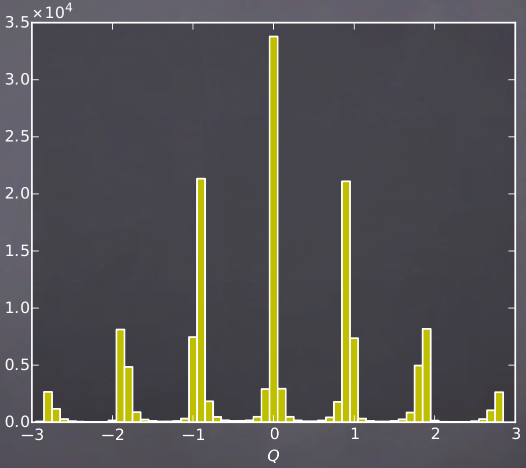
Topological charge distribution

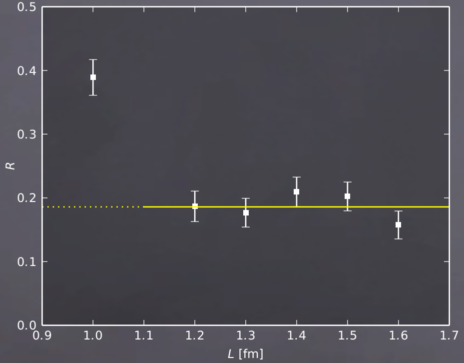


at flow-time $t = 0.0355\,\mathrm{fm}^2$

Lattices details

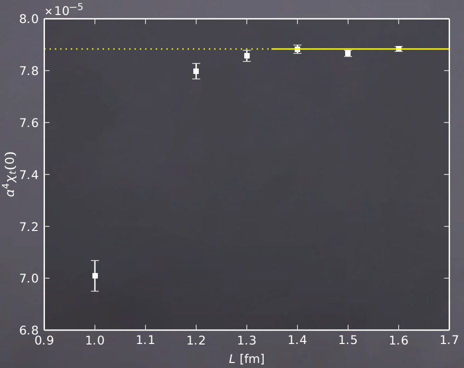
run	β	$\frac{L}{a}$	a[fm]	L[fm]	N_{conf}
A_1	5.96	10	0.102	1.0	36 000
B_1	5.96	12	0.102	1.2	144 000
C_1	5.96	13	0.102	1.3	280 000
D_1	5.96	14	0.102	1.4	505 000
E_1	5.96	15	0.102	1.5	880 000
F_1	5.96	16	0.102	1.6	1 500 000
B_1	5.96	12	0.102	1.2	144 000
B_2	6.05	14	0.088	1.2	144 000
B_3	6.13	16	0.077	1.2	144 000
B_4	6.21	18	0.068	1.2	144 000

(Preliminary) Results
Thermodynamic limit – R



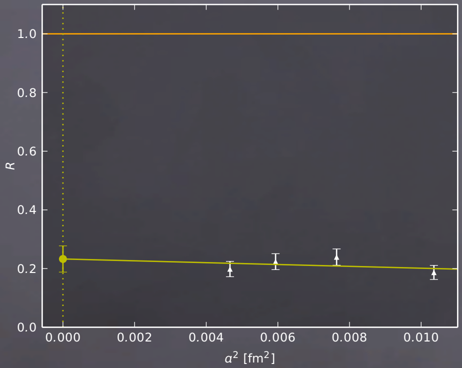
- Finite volume effects on R are compatible with statistical errors at $V = L^4 = (1.2\,\mathrm{fm})^4$
- Statistical error on R is $\mathcal{O}(V)$

(Preliminary) Results
Thermodynamic limit – χ_t^{YM}



- Finite volume effects on R are compatible with statistical errors at $V = L^4 = (1.2\,\mathrm{fm})^4$
- Statistical error on R is $\mathcal{O}(V)$
- The topological susceptibility χ_t^{YM} shows finite volume effects at $1.2\,\mathrm{fm}$

(Preliminary) Results
Continuum limit – R

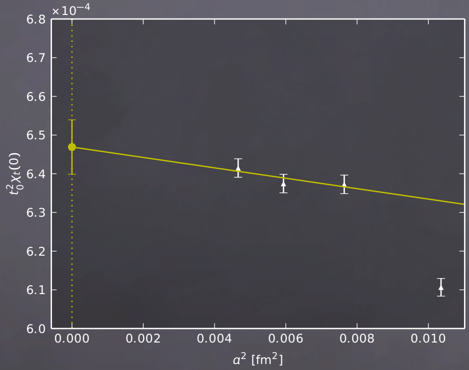


- Statistical error on R is $\mathcal{O}(V)$
- Continuum limit done at fixed volume $(1.2\,\mathrm{fm})^4$
- Continuum limit value

$$R = 0.233(45)$$

- Not compatible with dilute instanton gas prediction $R = 1$

(Preliminary) Results
Continuum limit – χ_t^{YM}



- Continuum limit value

$$t_0^2\chi_t^{\mathrm{YM}} = 6.47(7)\times 10^{-4}$$

- Result corrected against finite volume effects

$$t_0^2\chi_t^{\mathrm{YM}} = 6.54(8)\times 10^{-4}$$

$$r_0^4\chi_t^{\mathrm{YM}} = 0.0526(28)$$

$$\chi_t^{\mathrm{YM}} = (185(5)\,\mathrm{MeV})^4$$

Comparison with other results

Our (preliminary) results:

$$R = 0.233(45)$$

$$r_0^4 \chi_t^{\mathrm{YM}} = 0.0526(28), \chi_t^{\mathrm{YM}} = (185(5)\,\mathrm{MeV})^4$$

Results from previos lattice simulations
using Yang–Mills gradient flow:

- $r_0^4 \chi_t^{\mathrm{YM}} = 0.061(6)$, $\chi_t^{\mathrm{YM}} = (192(7)\,\mathrm{MeV})^4$ [Lüscher, Palombi 2010]

using Neuberger–Dirac operator:

- $r_0^4 \chi_t^{\mathrm{YM}} = 0.059(3)$, $\chi_t^{\mathrm{YM}} = (191(5)\,\mathrm{MeV})^4$
  [Del Debbio, Giusti, Pica 2005]
- $R = 0.30(11)$ [Giusti, Petrarca, Taglienti 2007]

Conclusions

The Witten–Veneziano mechanism links the η' mass with the topological charge distribution in $SU(3)$ YM theory

- We studied the topological charge distribution with unprecedented precision
- We implemented a new fourth-order integration method for the YM gradient flow
- First result of R with systematic and statistical errors under control: $R = 0.233(45)$
  - In agreement with Witten–Veneziano: $R = \mathcal{O}\left(\frac{1}{N_c^2}\right)$
  - Dilute instanton gas prediction $R = 1$ is inconsistent with this result
- Topological susceptibility χ_t^{YM} measured with unprecedented precision: $t_0^2 \chi_t^{\mathrm{YM}} = 6.53(8) \times 10^{-4}$
  - In agreement with previous results
  - Compatible with η' experimental mass as predicted by the Witten–Veneziano mechanism

Thanks
for your attention!

Backup

Runge–Kutta–Munthe-Kaas method

The YM gradient flow is the solution of the ordinary differential equation

$$\dot{V}(t) = Z[V(t)]V(t)$$

The fourth order Runge–Kutta–Munthe-Kaas method is

[Munthe-Kaas 1995; Munthe-Kaas 1998; Munthe-Kaas 1999]

$$\begin{aligned}
W_2 &= \exp\left\{\frac{1}{2}Z_1\right\}V(t) \\
W_3 &= \exp\left\{\frac{1}{2}Z_2 + \frac{1}{8}[Z_1, Z_2]\right\}V(t), \\
W_4 &= \exp\{Z_3\}V(t), \\
V(t+a^2\epsilon) &= \exp\left\{\frac{1}{6}Z_1 + \frac{1}{3}Z_2 + \frac{1}{3}Z_3 + \frac{1}{6}Z_4 - \frac{1}{12}[Z_1, Z_4]\right\}V(t).
\end{aligned}$$

where $Z_i = Z[W_i]$

Reference flow-time

The reference flow-time t_0 is defined by

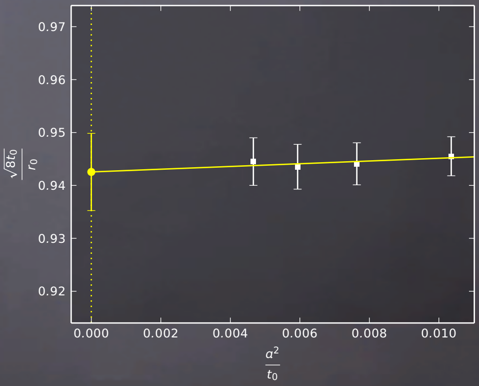
$$t^2 \langle E(t) \rangle \big|_{t=t_0} = 0.3$$

- Easy to measure with great statistical accuracy
- Our result

$$\frac{\sqrt{8t_0}}{r_0} = 0.943(7)$$

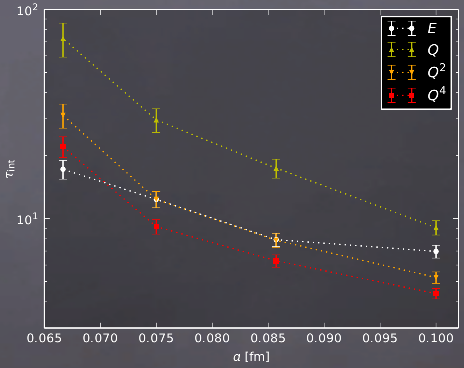
- using F_K measured in quenched QCD for scale setting

$$t_0 = 0.0290(16)\,\mathrm{fm}^2$$

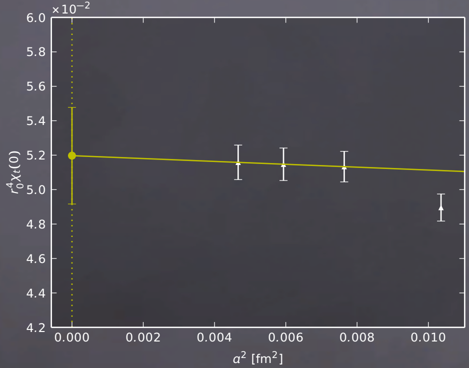


Autocorrelation

Integrated autocorrelation time τ_{int} for various observable versus the lattice spacing a



($\color{orange}{\text{Preliminary}}$) Results
Continuum limit – $r_0^4 \chi_t^{\mathrm{YM}}$



- Continuum limit value

$$r_0^4 \chi_t^{\mathrm{YM}} = 0.0520(28)$$

- Result corrected against finite volume effects

$$r_0^4 \chi_t^{\mathrm{YM}} = 0.0526(28)$$